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Boundary Element Solution of Heat Convection-Diffusion Problems

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A boundary element method is described in detail for the solution of two-dimensional steady-state convective heat diffusion problems in homogeneous and isotropic media with both linear and nonlinear boundary conditions. Through an exponential variable transformation, the introduction of fundamental solutions and the use of Green's theorem, the problem is reduced to one involving values of temperature and/or heat flux in the form of an integral only along the boundary. The integral is solved numerically for three examples. Two of them have linear boundary conditions and their numerical results are compared with the corresponding analytical solutions. The other has a nonlinear boundary condition due to heat radiation and an iterative procedure is applied to obtain the numerical solution. The fictitious source formulation leading to the boundary element solution of the same problems is discussed as an alternative. The extension of the method to formulate transient and/or three-dimensional convective heat diffusion problems is also described, and the relevant fundamental solutions are given. Finally, the exponential variable transformation is applied to construct a functional of variational principle which leads to developing a finite element formulation of the problems with a banded, symmetric stiffness matrix.

I. INTRODUCTION

Convective heat (and/or mass) transfer occurs as a basic phenomenon in many physical processes, such as single crystal growth [1], thin film growth [2], optical fiber processing [3–4], laser-assisted surface hardening [5], continuous casting [6], etc. In these processes, a knowledge of the distribution of field variables such as temperature or concentration is of vital importance to the understanding of the physics controlling the properties of the materials to be processed. Because of this importance, there have been continuous research efforts toward the more efficient and accurate prediction of the field distribution. In the case where the medium is fluid the complete mathematical description of the heat convection problems would be a set of equations composed of the Navier–Stokes equations, which govern the

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flow field, coupled with the Fourier heat conduction equation with convective terms included. However, in some situations [1–6], for example, where the workpiece is solid and moves as a whole at a constant velocity with reference to a fixed coordinate, the heat transfer mechanism can be described by heat conduction with a constant convective velocity, and hence the solutional algorithm should be greatly simplified.

Numerical algorithms developed to solve this type of problem can be classified, in general, into two categories, the domain method and the boundary method, with the former comprised of the finite difference and finite element methods and the latter comprised of the boundary element or boundary integral methods. All the algorithms, however, have one feature in common in that they start with the pure diffusion equation and then incorporate a special treatment of the convection term.

The approach adopted in the finite difference method is to approximate the partial differential operators by finite difference operators. Likewise, the convection term is approximated by either a forward-, central-, or backward difference scheme depending on the geometric locations under consideration [5, 6]. In the finite element approach, on the other hand, the weighted residue method or weak formulation is often used to formulate the problem, which usually results in an unsymmetric banded coefficient matrix because of the presence of the convection term. The solution of the resultant unsymmetric matrix is often computationally time-consuming, which is especially true when the nonlinear properties are to be accounted for. This has motivated some investigators [8] to treat the convection term as a "source term" in order to preserve the banded symmetry for the resultant matrix, so as to facilitate the use of efficient linear solvers. In doing so, however, an iterative procedure must be used.

In recent years, there has been increasing interest in using the boundary element method to solve virtually all kinds of applied continuum mechanics problems because of its potential to reduce problem dimensionality. The method formulates the problems using properties only along the boundaries of the problem domain through the use of Green's second identity and the corresponding fundamental solution or Green's function in infinite space [9–10]. This should be very attractive because using the boundary properties alone means the reduction of the number of unknowns to be solved for, thus saving computing time. Sometimes this approach is also considered as a generalization of weighted residual formulation with the fundamental solution regarded as a special weighting function [11].

Treatments of heat convection problems in this category are those of Skegart and Brebbia [12] and of Onishi *et al.* [13]. In their approaches, the convection term is treated as a "source" term and the fundamental solution to the Laplacian equation (or pure diffusion equation) is used. As a result, an iterative procedure has to be used to solve the resulting discretized equations. Moreover, the attraction of reducing problem dimensionality seems to disappear since the "source" is fully populated over the whole domain. The scheme is not particularly suitable for heat diffusion with a uniform motion as the problem could be otherwise entirely formulated only along the boundaries (see below). Nevertheless, their approaches are general in the sense that they are applicable to the problems of diffusion with both constant and/or varying flow field.

In this paper, a methodology for the boundary element (or boundary integral) solution of heat convection-diffusion problems based on an exponential transformation is described. While the transformation has been used elsewhere [14–15] for the purpose of generating analytical solutions, its use in assisting the formulation of boundary element solutions to problems involving both convection and diffusion has been largely overlooked and only a few papers have been published [16–18]. Ikeuchi *et al.* [16, 17] presented a constant boundary element solution of convective heat transfer in three dimensions with linear boundary conditions while Okamoto [18] derived a boundary element formulation of a chemical reaction system where both convective diffusion and chemical reaction occur. In what follows, the boundary element formulation and implementation of a two-dimensional convection–diffusion problem is first presented, followed by its application to three examples with both linear and nonlinear boundary conditions, and the numerical results are compared with the analytical solutions available. A fictitious source formulation leading to the boundary element solution is then discussed as an alternative approach. Also, a functional of variational principle is constructed through the application of the exponential transformation, which naturally puts the finite element formulation of this type of problem on the basis of variational principle and results in a banded, symmetric linear system. Finally, extension of the formulation and solution procedures to the solution of time-dependent and three-dimensional heat convection–diffusion problems is discussed and relevant fundamental solutions are presented.

II. Boundary Element Formulation and Numerical Implementation

A. *Problem and Formulation*

The problem of heat convection–diffusion to be considered is illustrated in Fig. 1, where a workpiece moves at a constant velocity V and alters its temperature distribution by exchanging heat with the surroundings. When formulated in Euler's coordinates, the problem takes the form

$$\nabla^2 T - 2\alpha(\hat{x}\cdot\nabla T) = 0, \qquad 0 \leqslant x \leqslant L, \qquad 0 \leqslant y \leqslant b, \tag{1}$$

where T is the temperature, x the unit vector of x-direction, and $\alpha \equiv V\rho c_P/2k$ in which ρ is the density, c_P the heat capacity, and k the thermal conductivity. The boundary conditions imposed are assumed to be of Dirichlet, Neumann, radiation, or mixed type.

The approach taken here to develop the boundary element formulation of the above problem involves reducing the problem to a Helmholtz equation through a variable transformation and subsequently solving the equation.

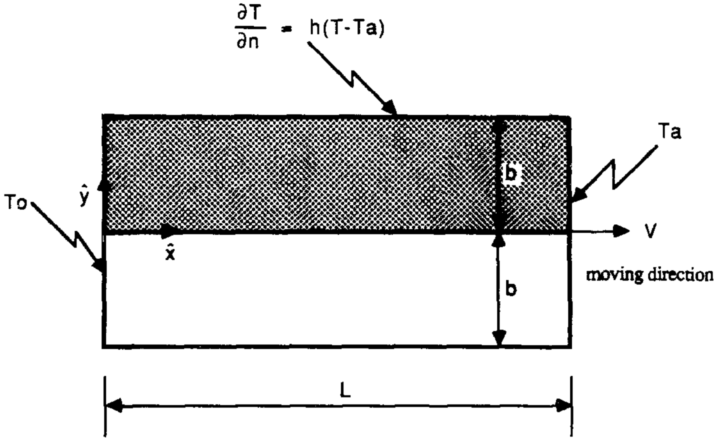

FIG. 1. Schematic representation of a convective heat diffusion problem.

The variable transformation has the form [14, 15]

$$T = \phi e^{\alpha x}. \tag{2}$$

Upon substituting it into Eq. (1) and rearranging, one obtains the Helmholtz equation as

$$\nabla^2 \phi - \alpha^2 \phi = 0, \tag{3}$$

where ϕ is an intermediate variable.

Note that Eq. (3) is similar to an equation of wave propagation but different from it in that the coefficient of the second term is a real number. This real number, in the terminology of wave scattering, indicates a decay of thermal "waves" associated with Eq. (3).

The fundamental solution corresponding to Eq. (3) should satisfy the equation

$$\nabla^2 G^* - \alpha^2 G^{\rceil} = -\delta(\mathbf{r} - \mathbf{r}') \tag{4}$$

and, in general, is expressible as [9],

$$G^*(\mathbf{r}, \mathbf{r}') = -\frac{i}{4} H_0^{(2)}(i\,|\alpha(\mathbf{r} - \mathbf{r}')|) \tag{5}$$

but can be cast in a more useful form [19], i.e.,

$$G^*(\mathbf{r}, \mathbf{r}') = \frac{1}{2\pi} K_0(|\alpha(\mathbf{r} - \mathbf{r}')|), \tag{6}$$

where $H_0^{(2)}$ is the Hankel function of the second kind of order zero, $i \equiv \sqrt{-1}$,

$|\mathbf{r}-\mathbf{r}'| = [(x-x')^2+(y-y')^2]^{1/2}$, and K_0 the modified Bessel function of the second kind of order zero.

The general procedures leading to the boundary integral formulation are illustrated as follows. Upon multiplying Eq. (3) by G^* and Eq. (4) by ϕ and subtracting, followed by integrating over the whole domain, one has

$$\int_\Omega \delta(\mathbf{r}-\mathbf{r}')\,\phi(\mathbf{r}')\,dv' = \int_\Omega (G^*\nabla^2\phi - \phi\nabla^2 G^*)\,dv'. \tag{7}$$

While the domain integral on the right-hand side can be reduced to a boundary integral by applying Green's second identity,

$$\int_\Omega (G^*\nabla^2\phi - \phi\nabla^2 G^*)\,dv = \oint_{\partial\Omega} (G^* q - \phi q^*)\,d\Gamma \tag{8}$$

that on the left-hand side can be integrated analytically by applying a limit procedure [20] when $\mathbf{r}$ lies on the boundary, viz.,

$$C(\mathbf{r})\,\phi(\mathbf{r}) = \int_\Omega \delta(\mathbf{r}-\mathbf{r}')\,\phi(\mathbf{r}')\,dv'. \tag{9}$$

In the above two equations, Ω represents the domain of interest, $\partial\Omega$ its boundary, q the derivative of ϕ, and q^* the derivative of the fundamental solution G^*,

$$q^* = \frac{\partial G^*}{\partial n'} = \frac{|\alpha|}{2\pi} K_1(|\alpha(\mathbf{r}-\mathbf{r}')|)\cdot\frac{(\mathbf{r}-\mathbf{r}')\cdot\mathbf{n}'}{|\mathbf{r}-\mathbf{r}'|}, \tag{10}$$

where K_1 is the modified Bessel function of the first kind of order one. The coefficient $C(\mathbf{r})$ generally takes the following values depending on the location of $\mathbf{r}$ [21]:

$$C(\mathbf{r}) = \begin{cases} 1 & \text{when } \mathbf{r} \text{ lies inside domain } \Omega \\ \frac{1}{2} & \text{when } \mathbf{r} \text{ lies on a smooth boundary } \partial\Omega \\ (\pi+\gamma_1-\gamma_2)/2\pi & \text{when } \mathbf{r} \text{ lies on a nonsmooth boundary } \partial\Omega, \end{cases}$$

where γ_1 and γ_2 are the angles between the outward normal of the non-smooth boundary and x-direction around a sharp corner point.

Upon substituting Eqs. (8) and (9) back into Eq. (7), followed by the inverse transformation of Eq. (2) and rearranging, one obtains the integral formulation of the convective diffusion problems using the field values only along the boundaries, i.e.,

$$C(\mathbf{r})\,T(\mathbf{r}) + \int_{\partial\Omega} T(\mathbf{r}')\,e^{-\alpha(x'-x)}\frac{\partial G^*}{\partial n'}\,d\Gamma(\mathbf{r}') + \int_{\partial\Omega} \alpha T(\mathbf{r}')\,e^{-\alpha(x'-x)}\frac{\partial x'}{\partial n'}G^*\,d\Gamma(\mathbf{r}')$$
$$= \int_{\partial\Omega} G^* e^{-\alpha(x'-x)}\frac{\partial T}{\partial n'}\,d\Gamma(\mathbf{r}'). \tag{11}$$

Note that the above boundary integral may be derived directly starting with Eq. (1) if its corresponding fundamental solution is regarded as

$$G^{**} = G^* e^{-\alpha(x' - x)}. \tag{12}$$

B. *Boundary Element Implementation*

The detailed procedures by which the boundary integral equation (12) is solved are documented in the literature [21, 22]. In general, they involve discretizing the relevant boundaries into small elements over each of which a polynomial interpolation function of unknowns is constructed, then evaluating the source–response coefficients between the boundary elements, and finally solving the resultant matrix equations for the unknowns. Polynomials of various degree may be used as an interpolation function over an element, but the choice is usually made based on a combination of the accuracy requirement and computational efficiency. In this study, a linear element is universally used in generating the results that are presented in the next section.

Following the discretization procedure and also making use of the isoparametric elements to represent the geometric variation of the boundaries, one can write the boundary element form of the integral equation for a point under consideration, $\mathbf{r}_i$, as

$$\begin{aligned} C(\mathbf{r}_i)\,T(\mathbf{r}_i) + \sum_{j=1}^{N} \Bigg\{ \int_{\partial\Omega_j(\xi)} [\psi^1(\xi), \psi^2(\xi)] \bigg(q^*(\mathbf{r}_i, \mathbf{r}'(\xi)) \\ + \alpha \frac{\partial x'}{\partial n'} G^*(\mathbf{r}_i, \mathbf{r}'(\xi)) \bigg) e^{-\alpha(x'(\xi) - x_i)} \, |J(\xi)| \, d\xi \Bigg\} \begin{Bmatrix} T_1^j \\ T_2^j \end{Bmatrix} \\ = \sum_{j=1}^{N} \Bigg\{ \int_{\partial\Omega_j(\xi)} [\psi^1(\xi), \psi^2(\xi)] \, G^*(\mathbf{r}_i, \mathbf{r}'(\xi)) \, e^{-\alpha(x'(\xi) - x_i)} \, |J(\xi)| \, d\xi \Bigg\} \begin{Bmatrix} q_1^j \\ q_2^j \end{Bmatrix}, \end{aligned} \tag{13}$$

where ψ_p $(p = 1, 2)$ is the interpolation function, $|J(\xi)|$ the Jacobian of the coordinate transformation, ξ the local coordinate, N the total number of boundary elements, and j the element number.

After applying Eq. (13) to all boundary nodes, one finally obtains a set of equations which, when written in matrix form, become

$$[\mathbf{H}]\{\mathbf{T}\} = [\mathbf{G}]\{\mathbf{q}\}, \tag{14}$$

where

$$G_{ij} = g_{i,j-1}^{(2)} + g_{i,j}^{(1)} \tag{15}$$

$$H_{ij} = \begin{cases} h_{i,j-1}^{(2)} + h_{i,j}^{(2)} & \text{for} \quad j \neq i \\ h_{i,j-1}^{(2)} + h_{i,j}^{(2)} + C_i & \text{for} \quad j = i \end{cases} \tag{16}$$

and $g_{i,j}^{(k)}$ and $h_{i,j}^{(k)}$ are of the form

$$g_{i,j}^{(k)} = \int_{\partial\Omega_j(\xi)} \psi_k(\xi)\, G^*(\mathbf{r}_i, \mathbf{r}'(\xi))\, e^{-\alpha(x'(\xi)-x_i)}\, |J(\xi)|\, d\xi \tag{17}$$

$$h_{i,j}^{(k)} = \int_{\partial\Omega_j(\xi)} \psi_k(\xi) \left(q^*(\mathbf{r}_i, \mathbf{r}'(\xi)) + \alpha \frac{\partial x'(\xi)}{\partial n'} G^*(\mathbf{r}_i, \mathbf{r}'(\xi)) \right) \times e^{-\alpha(x'(\xi)-x_i)}\, |J(\xi)|\, d\xi. \tag{18}$$

The above element integrals can be evaluated numerically using Gaussian quadrature when node i does not belong to the jth element. In the present study, the four-point integration rule is used. The modified Bessel functions of the second kind of order zero or one are approximated by the polynomials given by Abramowitz and Stegun [23].

When node i happens to lie on the jth element, however, the Gauchy principal value of the integrals must be taken because of the occurrence of singularity in the fundamental solution. Caution must be exercised in treating the singularity here as it plays a crucial role in determining the accuracy of the solution. Although some numerical quadrature rules have been developed for estimating the integration of singularities [21], they apply only to some simple cases and thus have limited use. As a rule of thumb, the term involving singularity should be analytically evaluated whenever possible in order to minimize errors in computation. Since the treatment of singularities encountered in the present study is tedious, only an outline is given below and the details are left in the Appendix. In the above equations, the integral involving q^* is zero as a consequence of orthogonality between the outward normal of the jth element and the direction of the integration path. Of the two g_{ij} terms, one can be shown to have a well-behaved integrand, thus permitting a numerical integration, and the other may be integrated analytically (see Appendix).

Before the matrix equation (14) is solved, the boundary conditions must be imposed. With the Dirichlet and Neumann boundary conditions, Eq. (14) can be rearranged such that

$$[\mathbf{K}]\{\mathbf{u}\} = \{\mathbf{F}\}, \tag{19}$$

where $\{\mathbf{u}\}$ is a set of unknowns of temperature or its derivative. Equation (19) is then solved by the standard Gaussian elimination method.

When part of the boundary is subject to boundary conditions of mixed type, i.e.,

$$\frac{\partial T}{\partial n} = d - eT, \tag{20}$$

the discretized matrix equations for this part of the boundary will be

$$[\mathbf{H}]\{\mathbf{T}\} = [\mathbf{G}]\{\mathbf{q}\} = [\mathbf{G}](\{\mathbf{D}\} - [\mathbf{E}]\{\mathbf{T}\}) \tag{21}$$

or

$$([\mathbf{H}]+[\mathbf{E}])\{\mathbf{T}\}=[\mathbf{G}]\{\mathbf{D}\} \tag{22}$$

which has the same form as Eqs. (19). In this case, after the system of Eqs. (19) is solved, the normal derivatives of potentials along that part of the boundary can be evaluated pointwise using Eq. (20).

When nonlinear boundary conditions (e.g., due to heat radiation) are considered, the coefficients d and e in Eq. (20) are temperature dependent and so are $[\mathbf{K}]$ and $\{\mathbf{F}\}$. Consequently, an iterative procedure must be employed to solve the system of the resulting nonlinear equations. There are available many algorithms [24, 25]. Some of them, which are based on the Newton–Raphson and/or optimization schemes and possess a relatively high convergence rate, have been successfully applied to solve nonlinear potential problems [26–29]. For the sake of illustration, a simple successive substitution procedure is adopted in this study, viz.,

$$[\mathbf{K}(\mathbf{T}_{i-1})]\{\mathbf{u}_i\}=\{\mathbf{F}(\mathbf{T}_{i-1})\}. \tag{23}$$

The iteration continues until the present tolerance, δ, on each individual u_j is met,

$$\left|\frac{u_j-u_{j-1}}{u_j}\right|\leqslant\delta. \tag{24}$$

It is noteworthy that, in contrast to divergence encountered with the use of the successive substitution in pure nonlinear diffusion problems [21], this study has found that convergence is well achieved within 10 iterations for the various cases tested (see below) for a specified $\delta=1\times10^{-4}$.

III. Numerical Applications

In this section, the algorithm described above is applied to study three examples. The first two examples have linear boundary conditions and the numerical results are compared with the analytical solutions. The numerical results for the third example, which has a nonlinear radiation boundary condition, are computed using the iterative procedure (Eq. (23)). In all the examples, the boundary is discretized into linear elements and the corners are rounded off as suggested by Brebbia *et al.* [21].

i. *A Long Bar*

The classical example of convection heat transfer problems, which is often used to test numerical algorithms, is the one-dimensional problem of a long bar moving at constant velocity, V, subject to $T=T_0$ at $x=0$ and $T=T_1$ at $x=L$ [30]. The numerical values adopted for the geometric dimensions, physical properties, and boundary values are $L=6$ m, $\alpha=-0.8$ to 3.0 (1/m), $T_0=300°$K, and $T_1=0°$K.

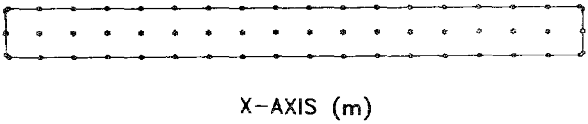

FIG. 2. Boundary discretization of the one-dimensional problem. Temperatures at points designated by * are plotted in Fig. 3.

The analytical solution to this problem is easily shown to be

$$\frac{T}{300}=e^{\alpha x}\frac{\sinh(\alpha(6-x))}{\sinh(6\alpha)}. \tag{25}$$

The bar is modeled numerically as two-dimensional with a height of 0.2 m and discretized using 42 equal linear boundary elements, as appears in Fig. 2. Numerical results are presented in Fig. 3 along with the analytical solution (Eq. (25)) for the above range of α. A very good agreement can be seen between the two. It should be stressed here that the match is still very good even within the region of the sharp front due to a large value of α. Furthermore, our additional numerical experiments have demonstrated that the boundary element solution is not prone to oscillation [illegible] method is superior to the domain based methods as both the finite difference and finite element solutions present oscillations, particularly near the sharp front, and thus special numerical schemes such as upwinding or streamline upwinding must be

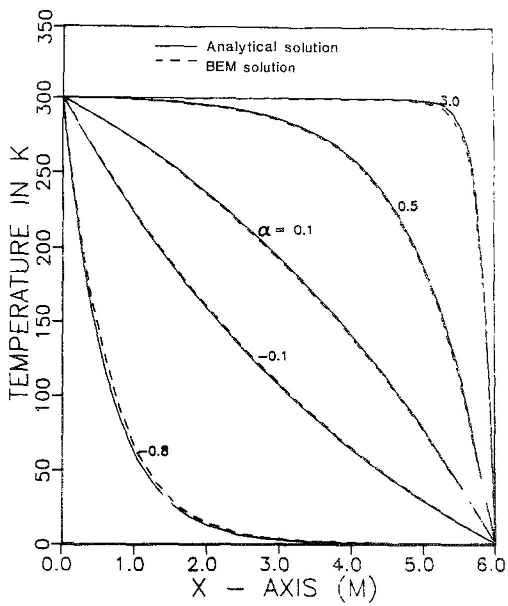

FIG. 3. Temperature distribution for various α values for the one-dimensional problem.

used to minimize this oscillatory behavior [30, 31]. Other studies [16, 17] also found that the boundary element solution, apart from its freedom from oscillations, possesses a higher accuracy than the finite element solution.

ii. *A Rectangular Plate with Linear Boundary Conditions*

This second example studies temperature distributions over a two-dimensional domain, as shown in Fig. 1, with the numerical values $\alpha = 0.15$ (1/m), $L = 6$ m, and $b = 4$ m and subject to the following boundary conditions:

$$T(x, -4) = T(x, 4) = 0 \tag{26}$$

$$T(0, y) = 0 \tag{27}$$

$$\frac{\partial T}{\partial y}(6, y) - \alpha T(6, y) = 100e^{6\alpha}. \tag{28}$$

The analytical solution to this problem can be obtained by the finite Fourier transformation after substituting $T = \phi e^{\alpha x}$ into the governing equation (Eq. (1)). The results are

$$T(x, y) = \frac{400}{\pi} e^{\alpha x} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n\beta_n} \frac{\sinh x\beta_n}{\cosh 6\beta_n} \sin\frac{n\pi(y+4)}{8}, \tag{29}$$

where $\beta_n^2 = \alpha^2 + n^2\pi^2/8^2$.

Figure 4 compares the analytical solution with numerical results obtained using

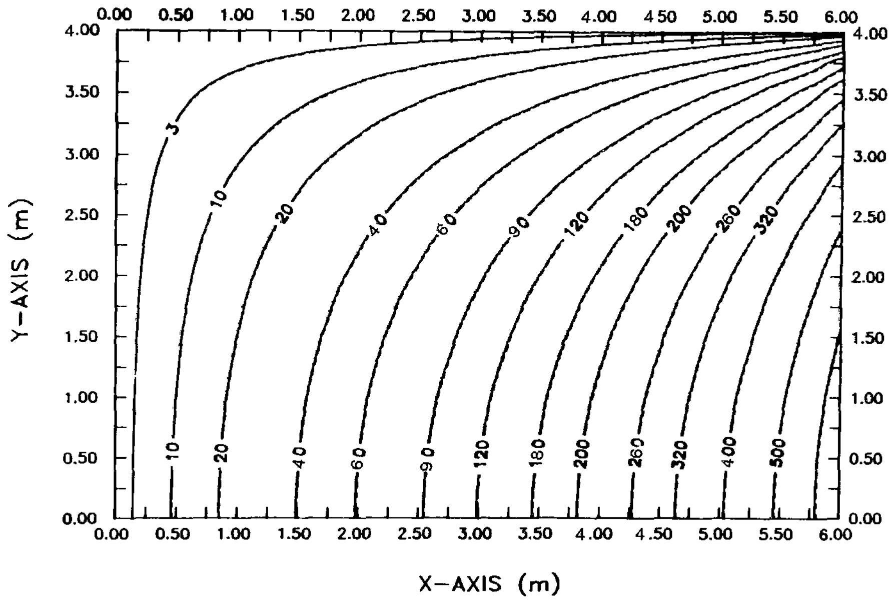

FIG. 4. Temperature distribution for the two-dimensional problem with linear boundary conditions. Analytical solutions are indicated by solid lines and numerical solutions by dashed lines.

84 linear boundary elements. Because of the symmetry with respect to $y=0$, only the upper half of the domain is used. It can be seen that the results agree very well.

iii. *The Rectangular Plate with Nonlinear Boundary Conditions*

In this example, the geometric dimensions and physical properties are the same as in the previous case. However, the boundary conditions on the upper and lower boundaries are relaxed so as to allow a combined heat transfer due to convection and radiation, i.e.,

$$q=h(T_a-T)-\sigma\varepsilon(T^4-T_a^4), \qquad y=\pm 4. \tag{30}$$

The other boundary conditions are

$$T(0, y)=T_0 \tag{31}$$

$$T(6, y)=T_a. \tag{32}$$

Again, only the upper half domain is used because of the symmetry of the problem. The whole boundary is discretized into 164 linear elements. The successful substitution iteration procedure, as described in Section II, is employed and convergence is achieved after eight iterations. Additional numerical experiments have also been conducted and the solutions are all converged within 10 iterations. The isothermal contours for the above problem are plotted in Fig. 5, where the following values are used: $T_a=273^\circ$K, $T_0=600^\circ$K, $\sigma\varepsilon=3.0\times 10^{-8}$ W/(m^{2}K^4), $h=0.25$ W/(m^{2}K), and $\delta=1\times 10^{-4}$. A three-dimensional view of the temperature distribution over the domain is given in Fig. 6. Inspection of Figs. 5 and 6 shows that while the tem-

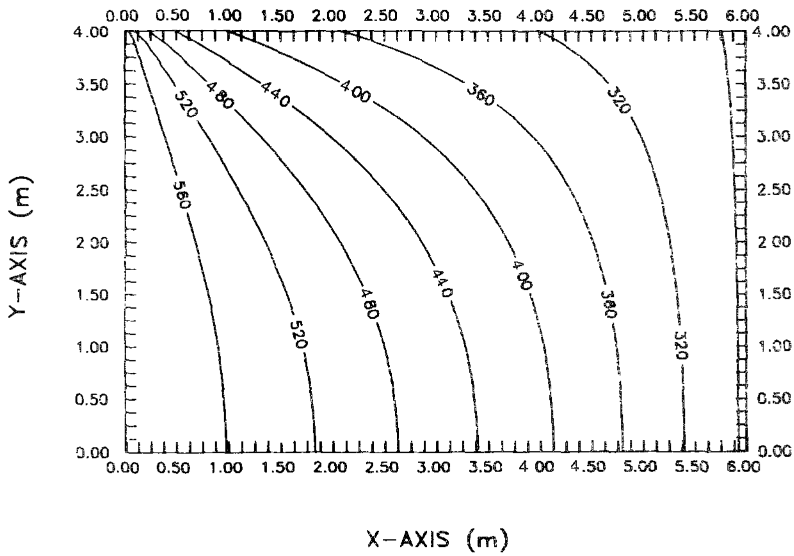

FIG. 5. Temperature distribution for the two-dimensional problem with nonlinear boundary conditions.

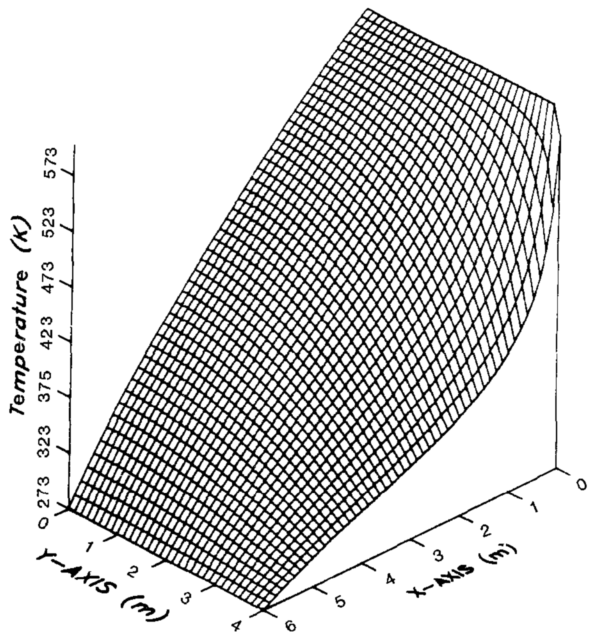

FIG. 6. Three-dimensional view of the temperature distribution for the nonlinear two-dimensional convective heat diffusion problem whose temperature conture is shown in Fig. 5.

perature along the center line ($y = 0$) changes almost linearly, that along the outer boundary ($y = 4$) varies dramatically because of the convective and radiative boundary condition, which is as expected.

IV. DISCUSSION

In the above, an efficient boundary element algorithm and its validation have been described. Several other related aspects of the boundary element solution of convective heat transfer problems warrant a separate discussion, however.

i. *Fictitious Source Formulation*

In Section IIA the general equations for the boundary element solution of convective diffusion problems are formulated; in Section IIB their implementation is described. An alternative to that implementation is obtained by introducing fictitious heat sources, a method used extensively in solving electrostatic or source-free magnetic problems [31].

In this approach, the scalar field, ϕ or T, is assumed to be due to the "heat charges" distributed along the boundary, i.e.,

$$\phi(\mathbf{r}) = \int_{\partial\Omega} \sigma_\phi(\mathbf{r}')\, G^*(\mathbf{r}, \mathbf{r}')\, d\Gamma(\mathbf{r}') \tag{33}$$

or

$$T(\mathbf{r}) = \int_{\partial\Omega} \sigma_T(\mathbf{r}')\, G^{**}(\mathbf{r}, \mathbf{r}')\, d\Gamma(\mathbf{r}'), \tag{34}$$

where σ is an unknown charge distribution on the domain boundary to be solved for by matching the boundary conditions imposed on variable ϕ or T; the subscripts represent the fields generated by the charges. Comparison of (33) with (34), together with the transformation $T = \phi e^{\alpha x}$, reveals that

$$\sigma_T(\mathbf{r}) = \sigma_\phi(\mathbf{r})\, e^{-\alpha x}. \tag{35}$$

The above formulation offers a major advantage in that the well-established programs for solving electrostatic or source-free magnetic problems can be easily modified (in most cases, only the boundary conditions need be changed and incorporated into the codes), to tackle the convective heat transfer problems, thus reducing programming effort.

ii. *Finite Element Formulation*

Although the transformation, $T = e^{\alpha x}$, has simplified the boundary element formulation, it can also be used to derive a functional so as to facilitate the finite element solution of the convective heat transfer problems expressed by Eq. (1).

In general, the finite element formulation of the problem is done through the weighted residue method or weak formation [7], which, in the presence of convective terms, generally does not obey the variational principle and results in a banded, unsymmetric stiffness matrix. However, starting with Eq. (2), it is straightforward to construct the variational integral as

$$J(T) = -\frac{1}{2}\int_{\Omega} e^{-2\alpha x}\, \nabla T \cdot \nabla T\, dV - \frac{1}{2}\int_{\partial\Omega} e^{-2\alpha x} h(T - T_s)^2\, d\Gamma \tag{36}$$

which corresponds to the convective heat transfer problem described by Eq. (1), subject to the natural boundary conditions

$$\frac{\partial T}{\partial n} = -h(T - T_s) \qquad \text{on } \partial\Omega. \tag{37}$$

This can be easily verified by substituting the functional (Eq. (36)) into the Euler–Lagrange differential equation [33]:

$$\frac{\partial J}{\partial T} - \nabla \cdot \left[\frac{\partial J}{\partial(\nabla J)}\right] = 0. \tag{38}$$

Application of the finite element solution procedure to Eq. (36) gives rise to a banded, symmetric matrix equation which can be solved very efficiently by available linear solvers.

iii. *Formulation of Time-Dependent Problems*

The procedures described in Section II can be naturally extended to formulate boundary element integrals of time-dependent convective heat transfer problems. With time as an additional variable, the differential equation becomes

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}-2\alpha\frac{\partial T}{\partial x}=D^{-1}\frac{\partial T}{\partial t}, \tag{39}$$

where D is the thermal diffusivity. Following the procedures in Section II, it is easy to show that

$$\begin{aligned} C(\mathbf{r})\,T(\mathbf{r},t)+\int_{t_0}^{t}\int_{\partial\Omega} T(\mathbf{r}',t)\,q^{**}(\mathbf{r},t;\mathbf{r}',\tau)\,d\Gamma(\mathbf{r}')\,d\tau \\ =\int_{t_0}^{t}\int_{\partial\Omega} q(\mathbf{r}',t)\,G^{**}(\mathbf{r},t;\mathbf{r}',\tau)\,d\Gamma(\mathbf{r}')\,d\tau \\ +\int_{\Omega} T(\mathbf{r}',t_0)\,G^{**}(\mathbf{r},t;\mathbf{r}',t_0)\,dV(\mathbf{r}'), \end{aligned} \tag{40}$$

where G^{**} corresponds to the fundamental solution of Eq. (39).

$$G^{**}(\mathbf{r},t;\mathbf{r}',\tau)=\frac{e^{-\alpha^2 D(t-\tau)}}{4D\pi(t-\tau)}\,e^{-\alpha(x'-x)}\,e^{-|\mathbf{r}-\mathbf{r}'|^2/4D(t-\tau)}. \tag{41}$$

The numerical solution of the integral equation (40) follows the same procedure as in Section II except that the domain integral term needs special treatment. The commonly used approach is to divide the domain into small internal cells over which the body integral is carried out [21, 34]. Having to discretize the whole domain has destroyed the feature of formulating problems only along the boundary. This has inspired some investigators to look for a boundary-alone method. Recently, the principle of dual reciprocity has been demonstrated to be very useful to formulate the time-dependent potential problems and interested readers are referred to Refs. [28, 35] for details.

iv. *Extension to Three-Dimensional Problems*

Thus far, the boundary element formulations and solution procedures have been discussed with respect only to two-dimensional convective heat transport problems. The solution of three-dimensional problems is, however, very similar and is described in detail elsewhere [16, 17] for a steady-state problem. The transient three-dimensional problems can be solved by following the procedure outlined in Section IV.iii and the corresponding fundamental solution needs to be used, which can be obtained by the integral transformation method [9]. For convenience, this fundamental solution is given as

$$G^{*}(\mathbf{r},t;\mathbf{r}',\tau)=\frac{e^{-\alpha^2 D(t-\tau)}}{[4D\pi(t-\tau)]^{3/2}}\,e^{-|\mathbf{r}-\mathbf{r}'|^2/4D(t-\tau)}. \tag{42}$$

V. Concluding Remarks

This paper has presented, in detail, a boundary element method for the solution of steady-state convection/diffusion problems in homogeneous and isotropic media with arbitrary (including nonlinear) boundary conditions. The method eliminates the common treatment of the convection part of the equation as a source term, as is required in other methods, through the use of the exponential variable transformation and thus permits the problem to be formulated using field variables such as temperature and/or heat flux only along the boundaries. Numerical solution procedures have been described with a special discussion of the singular integrals arising from the formulation. Two examples with linear boundary conditions have shown that the results obtained by this method compare very well with available analytical solutions. The method has been further illustrated by the third example which has a nonlinear boundary condition and is solved iteratively using the successive substitution technique. As a result of solving the problems only along the boundaries by this method, the number of unknowns is reduced in comparison with other methods in present use, thereby increasing computational efficiency.

Fictitious source formulation has been discussed as an alternative and the integral form of the solution has been given. This work has also extended the methodology to formulate the boundary integral solution of transient and/or three-dimensional convective heat diffusion problems and listed their relevant fundamental solutions. As a fringe benefit of the exponential variable transformation, a functional of variational principle has been constructed which leads naturally to the finite element formulation of the convection–diffusion problems with a resulting banded, symmetric stiffness matrix, thus permitting the use of computationally efficient linear solvers.

Appendix

Referring to Fig. A1, $x'(\xi)$ can be expressed as a function of local coordinate ξ, as

$$x'(\xi)=\tfrac{1}{2}(1-\xi)\,x'_1+\tfrac{1}{2}(1+\xi)\,x'_2=\tfrac{1}{2}(1-\xi)(x'_1-x'_2)+x'_2. \tag{A1}$$

By a variable transformation,

$$\mu=|\alpha J|\,(1-\xi); \tag{A2}$$

$g_{ij}^{(1)}$ can be expressed as

$$g_{ij}^{(1)}=\frac{1}{4\pi\alpha^2\,|J|}\,e^{-\alpha(x'_2-x)}\int_0^{2|\alpha J|}\mu e^{-\beta\mu}K_0(\mu)\,d\mu, \tag{A3}$$

where

$$\beta=\frac{(x'_1-x'_2)}{2\,|J|}. \tag{A4}$$

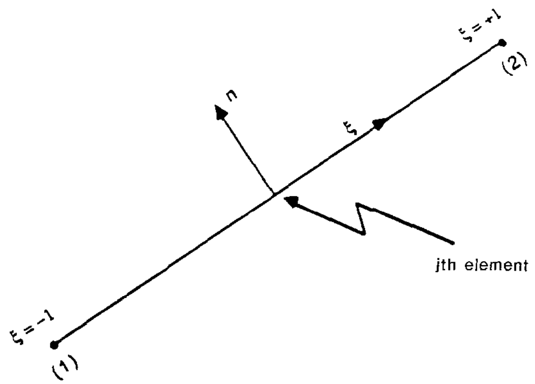

FIG. A1. A typical boundary element with a local coordinate system.

The integrand is well behaved and can be integrated numerically, since

$$\lim_{\mu \to 0} \mu e^{-\beta\mu} K_0(\mu) = \lim_{\mu \to 0} -\mu e^{-\beta\mu} \ln \mu = 0. \tag{A5}$$

It is a simple matter to show that

$$g_{ij}^{(2)} = \frac{1}{2\pi\,|\alpha|}\, e^{-\alpha(x_2' - x)} \int_0^{2\,|\alpha J|} e^{-\beta\mu} K_0(\mu)\, d\mu - g_{ij}^{(1)}. \tag{A6}$$

The integrand, $e^{-\beta\mu}K_0(\mu)$, is not well behaved, however, and possesses a logarithmic singularity. The integration can be evaluated analytically for some special cases [19], i.e.,

$$\int_0^{2\,|\alpha J|} e^{-\beta\mu} K_0(\mu)\, d\mu = 2\,|\alpha J|\, e^{-2\beta\,|\alpha J|}(K_0(2\,|\alpha J|) + K_1(2\,|\alpha J|)) \pm 1 \qquad (\beta = \pm 1) \tag{A7}$$

and

$$\begin{aligned}\int_0^{2\,|\alpha J|} & e^{-\beta\mu} K_0(\mu)\, d\mu \\ &= -(\gamma + \ln|\alpha J|)(2\,|\alpha J|) \sum_{k=0}^{\infty} \frac{(|\alpha J|)^{2k}}{(k!)^2\,(2k+1)} \\ &\quad + (2\,|\alpha J|) \sum_{k=0}^{\infty} \frac{(|\alpha J|)^{2k}}{(k!)^2\,(2k+1)^2} \\ &\quad + (2\,|\alpha J|) \sum_{k=1}^{\infty} \frac{(|\alpha J|)^{2k}}{(k!)^2\,(2k+1)} \left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right) \qquad (\beta = 0),\end{aligned} \tag{A8}$$

where γ is the Euler constant and $\gamma = 0.5772156649 \cdots$.

Note that the integral appearing in $g_{ij}^{(1)}$ can also be evaluated analytically when $\beta = 0$ [19].

$$\int_0^{2|\alpha J|} \mu K_0(\mu)\, d\mu = -2\,|\alpha J|\, K_1(2\,|\alpha J|) + 1. \tag{A9}$$

For an arbitrary β, the following step can be taken to evaluate the first term in Eq. (A6):

$$\int_0^{2|\alpha J|} e^{-\beta\mu} K_0(\mu)\, d\mu = \int_0^{2|\alpha J|} (e^{-\beta\mu} - 1)\, K_0(\mu)\, d\mu + \int_0^{2|\alpha J|} K_0(\mu)\, d\mu. \tag{A10}$$

While the last term in the above equation is nothing but Eq. (A8), the first term can be numerically integrated as the integrand is now well behaved, i.e.,

$$\lim_{\mu \to 0} (e^{-\beta\mu} - 1)\, K_0(\mu) = \lim_{\mu \to 0} \beta\mu \ln \mu = 0. \tag{A11}$$

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